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Strict Positivity of a Solution to a One-Dimensional Kac Equation Without Cutoff

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We consider the solution of a one-dimensional Kac equation without cutoff built by Graham and Méléard. Recalling that this solution is the density of a Poisson driven nonlinear stochastic differential equation, we develop Bismut's approach of the Malliavin calculus for Poisson functionals in order to prove that this solution is strictly positive on $]0, \infty[\times\mathbb{R}$.

KEY WORDS: Boltzmann equation without cutoff; Poisson measure; stochastic calculus of variations.

INTRODUCTION

We prove by a probabilistic approach the strict positivity of a solution of a one dimensional Kac equation without cutoff, in the case where the cross section does sufficiently explode. In the cutoff case, much more is known: Pulverenti and Wennberg,[14] have proved, by using analytic methods, the existence of a Maxwellian lowerbound. But their proof is based on the separation of the gain and loss terms, which typically cannot be done in the present case. Let us also mention that similar results about the Laudau equation, obtained by analytic methods, can be found in Arsen'ev, Buryak,[2] and Villani.[16] But no result seems to have been found by the analysts in the case of the Boltzmann or Kac equation without cutoff.

The solution we study has been built by Graham and Méléard in ref. 11, who follow the ideas of Tanaka,[15] and use the Malliavin calculus. This solution $f(t, v)$ can be related with the solution V_t of a Poisson driven nonlinear S.D.E.: for each $t>0$, $f(t, \cdot)$ is the density of the law of V_t. We will thus study f as the density of a Poisson functional.

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The strict positivity of the density for Wiener functionals has been worked out by Aida, Kusuoka, Stroock,[1] and Ben Arous, Léandre,[4] see also Bally, Pardoux.[3] In ref. 10, the strict positivity of the density for Poisson driven S.D.Es is studied in the case where the intensity measure of the Poisson measure is the Lebesgue measure. The method is adapted from a work of Bally and Pardoux,[3] which deals with a similar problem in the case of white noise driven S.P.D.E.s, i.e., with Wiener functionals. This method is based on Bismut's approach of the Malliavin calculus, which consists in perturbing the processes, see, e.g., Bichteler, Jacod,[5] for the case of classical diffusion processes with jumps. Nevertheless, we can not directly apply the results of ref. 10. We can not either use exactly the same Malliavin calculus as Bichteler and Jacod, because the intensity measure of our Poisson measure will not be the Lebesgue measure. We generalize a Malliavin calculus adapted to our model, inspired by Graham and Méléard.[11]

Let us say a word about the difference between the techniques in the case of Wiener functionals and Poisson functionals. The main difference is that the Malliavin calculus does product integrals with respect to the Lebesgue measure in the first case, and with respect to the Poisson measure in the second case. We thus have to deal with random perturbations and with stopping times instead of deterministic perturbations and times. This is why the assumptions are very stringent in ref. 10. Nevertheless, the method gives a quite good result in the case of the Kac equation without cutoff.

The present work is organized as follows. In Section 1, we recall the Kac equation, we give the results of Desvillettes, Graham, and Méléard in refs. 8 and 11, who solved this equation, and we state our result. In Section 2, we define rigorously our "perturbations," and we state a criterion of strict positivity. At last, we apply this criterion in the next sections.

1. THE KAC EQUATION WITHOUT CUTOFF, THE MAIN RESULT

The Kac equation deals with the density of particles in a gaz. We denote by $f(t, v)$ the density of particles which have the velocity $v \in \mathbb{R}$ at the instant $t > 0$. Then

$$\frac{\partial f}{\partial t}(t, v) = \int_{v_* \in \mathbb{R}} \int_{\theta = -\pi}^{\pi} [f(t, v')\, f(t, v'_*) - f(t, v)\, f(t, v_*)]\, \beta(\theta)\, d\theta\, dv_* \tag{1.1}$$

where

$$v' = v \cos\theta - v_* \sin\theta; \qquad v'_* = v \sin\theta + v_* \cos\theta \tag{1.2}$$

and β is a non cutoffed cross section, i.e., an even and positive function on $[-\pi, \pi]\backslash\{0\}$ satisfying

$$\int_0^\pi \theta^2 \beta(\theta)\, d\theta < \infty \tag{1.3}$$

The case with cutoff, namely when $\int_0^\pi \beta(\theta)\, d\theta < \infty$, has been much investigated by the analysts, and they have obtained some existence, regularity and strict positivity results.

In refs. 8 and 11, Desvillettes, Graham and Méléard give an existence and regularity result for such an equation, by using the probability theory. See also Desvillettes,[6] for another statement (using the Fourier Theory), and Desvillettes[7] or Fournier[9] for the 2-dimensional case. We are interested in this paper in the strict positivity of the solution of (1.1) built by Graham and Méléard in ref. 11. Let us recall their main results.

First, we will consider solutions in the following (weak) sense.

Definition 1.1. Let P_0 be a probability on $\mathbb{R}$ that admits a moment of order 2. A positive function f on $\mathbb{R}^+ \times \mathbb{R}$ is a solution of (1.1) with initial data P_0 if for every test function $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned}\int_{v\in\mathbb{R}} & f(t,v)\,\phi(v)\,dv \\ &= \int_{v\in\mathbb{R}} \phi(v)\,P_0(dv) + \int_0^t \int_{v\in\mathbb{R}} \int_{v^*\in\mathbb{R}} K^\phi(v, v_*)\, f(s,v)\, f(s,v^*)\,dv\,dv^*\,ds\end{aligned} \tag{1.4}$$

where

$$\begin{aligned}K^\phi(v, v_*) = -bv\phi'(v) + \int_{-\pi}^{\pi} &\{\phi(v\cos\theta - v_*\sin\theta) - \phi(v) \\ &-[v(\cos\theta - 1) - v_*\sin\theta]\,\phi'(v)\}\,\beta(\theta)\,d\theta\end{aligned} \tag{1.5}$$

and

$$b = \int_{-\pi}^{\pi} (1-\cos\theta)\,\beta(\theta)\,d\theta \tag{1.6}$$

Notice that b and the collision kernel K^ϕ are well defined thanks to (1.3).

In refs. 8 and 11, one assumes that

Assumption (H):

1. The initial data P_0 admits a moment of order 2, and is not a Dirac mass at 0.

2. $\beta = \beta_0 + \beta_1$, where β_1 is even and positive on $[-\pi, \pi] \backslash \{0\}$, and there exists $k_0 > 0$, $\theta_0 \in \,]0, \pi/2[$, and $r \in \,]1, 3[$ such that $\beta_0(\theta) = (k_0/|\theta|^r)\, 1_{[-\theta_0, \theta_0]}(\theta)$. We still assume $\int_0^\pi \theta^2 \beta(\theta)\, d\theta < \infty$.

They also build the following random elements:

Notation 1.2. We denote by N_0 and N_1 two independant Poisson measures on $[0, T] \times [0, 1] \times [-\pi, \pi]$, with intensity measures:

$$\nu_0(d\theta, d\alpha, ds) = \beta_0(\theta)\, d\theta\, d\alpha\, ds; \qquad \nu_1(d\theta, d\alpha, ds) = \beta_1(\theta)\, d\theta\, d\alpha\, ds \tag{1.7}$$

and by $\tilde{N}_0$ and $\tilde{N}_1$ the associated compensated measures. We will write $N = N_0 + N_1$. We consider a real valued random variable V_0 independant of N_0 and N_1, of which the law is P_0. We also assume that our probability space is the canonical one associated with the independent random elements V_0, N_0, and N_1:

$$\begin{aligned}(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P) = (\Omega', \mathscr{F}', \{\mathscr{F}'\}, P') &\otimes (\Omega^0, \mathscr{F}^0, \{\mathscr{F}^0_t\}, P^0)\\ &\otimes (\Omega^1, \mathscr{F}^1, \{\mathscr{F}^1_t\}, P^1)\end{aligned} \tag{1.8}$$

We will consider $[0, 1]$ as a probability space, denote by $d\alpha$ the Lebesgue measure on $[0, 1]$, and denote by E_α and $\mathscr{L}_\alpha$ the expectation and law on $([0, 1], \mathscr{B}([0, 1]), d\alpha)$.

The following theorem is proved in ref. 8 (Theorem 3.6, p. 11).

Theorem 1.3. There exists a process $\{V_t(\omega)\}$ on Ω and a process $\{W_t(\alpha)\}$ on $[0, 1]$ such that (b is defined by (1.6))

$$\left.\begin{aligned}&V_t(\omega) = V_0(\omega) + \int_0^t \int_0^1 \int_{-\pi}^{\pi} [(\cos\theta - 1)\, V_{s-}(\omega) - (\sin\theta)\, W_{s-}(\alpha)]\\ &\qquad\qquad \times \tilde{N}(\omega, d\theta\, d\alpha\, ds) - b \int_0^t V_{s-}(\omega)\, ds\\ &\mathscr{L}_\alpha(W) = \mathscr{L}(V); \qquad E(\sup_{[0,T]} V_t^2) < \infty\end{aligned}\right\} \tag{1.9}$$

At last, Graham and Méléard show in ref. 11 the following theorem (see Theorem 1.6, Corollary 1.8, p. 4)

Theorem 1.4. Assume (H). Let (V, W) be a solution of (1.9). Then for all $t > 0$, the law of V_t admits a density $f(t, \cdot)$ with respect to the Lebesgue measure on $\mathbb{R}$. The obtained function f is a solution of the Kac equation (1.1) in the sense of Definition 1.1. Assume furthermore that P_0 admits some moments of all orders. Then for each $t > 0$, the function $f(t, \cdot)$ is of class C^∞ on $\mathbb{R}$.

Let us now give our assumption, which is more stringent than (H): we need a stronger explosion of the cross section.

Assumption (SP):

1. The initial data P_0 admits moments of all orders, and is not a Dirac mass at 0.

2. $\beta = \beta_0 + \beta_1$, where β_1 is even and positive on $[-\pi, \pi]\backslash\{0\}$, and there exists $k_0 > 0$, $\theta_0 \in]0, \pi/2[$, and $r \in]2, 3[$ such that $\beta_0(\theta) = (k_0/|\theta|^r)\, 1_{[-\theta_0, \theta_0]}(\theta)$. We still assume $\int_0^\pi \theta^2 \beta(\theta)\, d\theta < \infty$.

Our result is the following:

Theorem 1.5. Assume (SP), and consider the solution in the sense of Definition 1.1 of Eq. (1.1) built in Theorem 1.4. Then f is strictly positive on $]0, +\infty[\times \mathbb{R}$.

In (SP), we do not really need the fact that P_0 has moments of all orders, but only the fact that the density $f(t, v)$ of the law of V_t built in Theorem 1.4 is continuous on $\mathbb{R}$ for each $t > 0$.

Notice that our method does not work in the case where r belongs to $]1, 2[$: we do really need a large explosion of the cross section at 0.

In the whole work, we will assume (SP), use Notation 1.2, and consider a solution (V, W) of (1.9).

2. A CRITERION OF STRICT POSITIVITY

This section contains two parts. We first introduce some general notations and definitions about Bismut's approach of the Malliavin calculus on our Poisson space. We follow here Bichteler, Jacod,[(5)] Graham and Méléard.[(11)] Then we adapt the criterion of strict positivity of Bally, Pardoux,[(3)] (which deals with the Wiener functionals) to our probability space.

Definition 2.1. A predictable function $v(\omega, s, \theta, \alpha)$ on $\Omega \times [0, T] \times [-\theta_0, \theta_0] \times [0, 1]$ is said to be a "perturbation" if for all fixed ω, s, α, $v(\omega, s, ., \alpha)$ is C^1 on $[-\theta_0, \theta_0]$, and if there exists some even positive (deterministic) functions η and ρ on $[-\theta_0, \theta_0]$ such that

$$|v(s, \theta, \alpha)| \leqslant \eta(\theta); \qquad |v'(s, \theta, \alpha)| \leqslant \rho(\theta) \tag{2.1}$$

$$\eta(\theta) \leqslant \frac{|\theta|}{2}; \qquad \eta(-\theta_0) = \eta(\theta_0) = 0 \tag{2.2}$$

$$\text{if} \quad \xi(\theta) = \rho(\theta) + r2^{r+2}\,\frac{\eta(\theta)}{|\theta|} \quad \text{then} \quad \|\xi\|_\infty \leqslant \frac{1}{2} \quad \text{and} \quad \xi \in L^1(\beta_0(\theta)\, d\theta) \tag{2.3}$$

Notice that thanks to (2.3), η and ρ are in $L^1 \cap L^\infty(\beta_0(\theta)\, d\theta)$.

Consider now a fixed perturbation v. For $\lambda \in [-1, 1]$ we set

$$\gamma^\lambda(s, \theta, \alpha) = \theta + \lambda v(s, \theta, \alpha) \tag{2.4}$$

Thanks to (2.1), (2.2), and (2.3), it is easy to check that for each λ, s, α, ω, $\gamma^\lambda(s, \cdot, \alpha)$ is an increasing bijection from $[-\theta_0, \theta_0] \backslash \{0\}$ into itself. Then we denote by $N_0^\lambda = \gamma^\lambda(N_0)$ the image measure of N_0 by γ^λ: for any Borel subset A of $[0, T] \times [-\theta_0, \theta_0] \times [0, 1]$,

$$N^\lambda(A) = \int_0^T \int_0^1 \int_{-\pi}^{\pi} 1_A(s, \gamma^\lambda(s, \theta, \alpha), \alpha)\, N_0(d\theta\, d\alpha\, ds) \tag{2.5}$$

We also define the shift S^λ on Ω by

$$V_0 \circ S^\lambda = V_0; \qquad N_0 \circ S^\lambda = N_0^\lambda; \qquad N_1 \circ S^\lambda = N_1 \tag{2.6}$$

We will need the following predictable function:

$$Y^\lambda(s, \theta, \alpha) = \frac{\beta_0(\gamma^\lambda(s, \theta, \alpha))}{\beta_0(\theta)}\,(1 + \lambda v'(s, \theta, \alpha)) \tag{2.7}$$

Then it is easy to check that for all $\lambda \in [-1, 1]$,

$$\gamma^\lambda(Y^\lambda \cdot v_0) = v_0 \tag{2.8}$$

and for all $\lambda, \mu \in [-1, 1]$ (recall that ξ is defined in (2.3)),

$$|Y^\lambda(s, \theta, \alpha) - Y^\mu(s, \theta, \alpha)| \leqslant |\lambda - \mu| \times \xi(\theta) \tag{2.9}$$

In order to check (2.9), it suffices to use on one hand

$$\left|\frac{\beta_0(\gamma^\lambda(s,\theta,\alpha))-\beta_0(\gamma^\mu(s,\theta,\alpha))}{\beta_0(\theta)}\right|$$

$$\leqslant \frac{1}{\beta_0(\theta)}\,|(\lambda-\mu)\,v(s,\theta,\alpha)| \sup_{\phi\in[\gamma^\lambda(s,\theta,\alpha),\,\gamma^\mu(s,\theta,\alpha)]} |\beta_0'(\phi)|$$

then the explicit expression of β_0, β_0', the fact that if $\theta>0$ (resp. $\theta<0$), then $[\gamma^\lambda(s,\theta,\alpha),\ \gamma^\mu(s,\theta,\alpha)]\subset\,]0,\pi]$, (resp. $[\gamma^\lambda(s,\theta,\alpha),\ \gamma^\mu(s,\theta,\alpha)]\subset[-\pi,0[$), and on the other hand

$$|\beta_0(\gamma^\lambda(s,\theta,\alpha))|\leqslant|\beta_0(\theta)|+|\beta_0(\gamma^\lambda(s,\theta,\alpha))-\beta_0(\gamma^0(s,\theta,\alpha))|$$

then the same computation as above.

We also consider the following martingale

$$M_t^\lambda=\int_0^t\int_0^1\int_{-\pi}^{\pi}(Y^\lambda(s,\theta,\alpha)-1)\,\tilde{N}_0(d\theta\,d\alpha\,ds) \tag{2.10}$$

and its Doléans-Dade exponential (see Jacod and Shiryaev,[13])

$$G_t^\lambda=\mathscr{E}(M^\lambda)_t=e^{M_t^\lambda}\prod_{0\leqslant s\leqslant t}(1+\Delta M_s^\lambda)\,e^{-\Delta M_s^\lambda} \tag{2.11}$$

Since $|Y^\lambda-1|\leqslant\xi\leqslant 1/2$, it is clear that G^λ is strictly positive on $[0,T]$ a.s. We now set $P^\lambda=G_T^\lambda.P$. Using Eq. (2.8), and the Girsanov Theorem for random measures (see Jacod and Shiryaev,[13] p. 157) one can show that $P^\lambda\circ(S^\lambda)^{-1}=P$, i.e., that the law of (V_0,N_0^λ,N_1) under P^λ is the same as the one of (V_0,N_0,N_1) under P.

We at last check the following lemma:

Lemma 2.2. Let v be a perturbation, and G^λ the associated exponential martingale. Then a.s., the map $\lambda\mapsto G_T^\lambda$ is continuous on $[-1,1]$.

Proof. Since $|Y^\lambda-1|\leqslant\xi\in L^1(\beta_0(\theta)\,d\theta)$, the compensated integrals can be splitted, and one obtains

$$\begin{aligned}G_T^\lambda=&\exp\left\{-\int_0^T\int_0^1\int_{-\pi}^{\pi}(Y^\lambda(s,\theta,\alpha)-1)\,\beta_0(\theta)\,d\theta\,d\alpha\,ds\right\}\\&\times\exp\left\{\int_0^T\int_0^1\int_{-\pi}^{\pi}\ln Y^\lambda(s,\theta,\alpha)\,N_0(d\theta\,d\alpha\,ds)\right\}\end{aligned} \tag{2.12}$$

Thanks to (2.9), it is clear that the first term in the product is continuous. Furthermore, we deduce from (2.9) and the fact that $\xi \leqslant 1/2$ that for all λ, μ,

$$|\ln Y^\lambda(s,\theta,\alpha) - \ln Y^\mu(s,\theta,\alpha)| \leqslant 2\,|Y^\lambda(s,\theta,\alpha) - Y^\mu(s,\theta,\alpha)| \leqslant 2\,|\lambda-\mu| \times \xi(\theta) \tag{2.13}$$

Since ξ is in $L^1(\beta_0(\theta)\,d\theta)$ the random variable $\int_0^T \int_0^1 \int_{-\pi}^{\pi} \xi(\theta)\, N_0(d\theta\, d\alpha\, ds)$ is a.s. finite, hence

$$\int_0^T \int_0^1 \int_{-\pi}^{\pi} \ln Y^\lambda(s,\theta,\alpha)\, N_0(d\theta\, d\alpha\, ds) \tag{2.14}$$

is a.s. Lipschitz on $[-1,1]$, and the second term in (2.12) is also continuous. The lemma is proved.

We now give the criterion of strict positivity we will use.

Theorem 2.3. Let X be a real valued random variable on Ω, such that $P \circ X^{-1} = p(x)\,dx$, with p continuous on $\mathbb{R}$, and let $y_0 \in \mathbb{R}$. Assume that there exists a sequence v_n of perturbations such that, if $X^n(\lambda) = X \circ S_n^\lambda$, then for all n, the map

$$\lambda \mapsto X^n(\lambda) \tag{2.15}$$

is a.s. twice differentiable on $[-1,1]$. Assume that there exists $c>0$, $\delta>0$, and $k<\infty$, such that for all $r>0$,

$$\lim_{n\to\infty} P(\Lambda^n(r)) > 0 \tag{2.16}$$

where

$$\Lambda^n(r) = \left\{ |X - y_0| < r, \left|\frac{\partial}{\partial\lambda} X^n(0)\right| \geqslant c, \sup_{|\lambda|\leqslant\delta} \left[\left|\frac{\partial}{\partial\lambda} X^n(\lambda)\right| + \left|\frac{\partial^2}{\partial\lambda^2} X^n(\lambda)\right| \right] \leqslant k \right\} \tag{2.17}$$

Then $p(y_0) > 0$.

In order to prove this criterion, we will use the following uniform local inverse theorem, that can be found in Aida, Kusuoka, and Stroock.[1]

Lemma 2.4. Let $c>0$, $\delta>0$, and $k<\infty$ be fixed. Consider the following set:

$$\mathscr{G}=\{g\colon \mathbb{R}\mapsto\mathbb{R}/|g'(0)|\geqslant c,\ \sup_{|x|\leqslant\delta}\ [|g(x)|+|g'(x)|+|g''(x)|]\leqslant k\} \tag{2.18}$$

Then there exists $\alpha>0$ and $R>0$ such that for every $g\in\mathscr{G}$, there exists a neigbourhood $\mathscr{V}_g$ of 0 contained in $]-R,R[$ such that g is a diffeomorphism from $\mathscr{V}_g$ to $]\,g(0)-\alpha,\ g(0)+\alpha[$.

Since this lemma deals with the behaviours near 0, it can obviously be adapted to functions from $[-1,1]$ to $\mathbb{R}$.

Proof of Theorem 2.3. Step 1. First notice that for all $r\leqslant 1$, for all n, and all $\omega\in\Lambda^n(r)$,

$$\sup_{|\lambda|\leqslant\delta}\ |X^n(\omega,\lambda)|\leqslant|X^n(\omega,0)|+\delta k=|X(\omega)|+\delta k\leqslant|y_0|+1+\delta k=k' \tag{2.19}$$

Thus, using Lemma 2.4, there exists $\alpha>0$ and $R\in\,]0,1]$ (depending only on δ, c, k, and k'), such that for all $r\leqslant 1$, all $n\in\mathbb{N}$, and all $\omega\in\Lambda^n(r)$, there exists $V_n(\omega)$ a neighbourhood of 0 contained in $]-R,R[$ such that the map

$$\lambda\mapsto X^n(\omega,\lambda) \tag{2.20}$$

is a diffeomorphism from $V_n(\omega)$ to $]X^n(\omega,0)-\alpha,\ X^n(\omega,0)+\alpha[\ =\ ]X(\omega)-\alpha,\ X(\omega)+\alpha[$.

Choosing α small enough, we can assume that $R\leqslant c/2k$. Thus, for all $\omega\in\Lambda^n(r)$ and $\lambda\in V_n(\omega)$, we have $|(\partial/\partial\lambda)\ X^n(\lambda)|\geqslant c/2$.

We now fix $r<\alpha$, and choose n large enough such that $P(\Lambda^n(r))>0$.

Step 2. The perturbations have been built in order to obtain, for all λ and all $f\in C_b^+(\mathbb{R})$,

$$E(f(X))=E(f(X^n(\lambda))\ G_T^n(\lambda)) \tag{2.21}$$

Thus

$$\begin{aligned}E(f(X))&=\tfrac{1}{2}\int_{-1}^{1}E(f(X^n(\lambda))\ G_T^n(\lambda))\ d\lambda\\&\geqslant\tfrac{1}{2}E\left[\int_{V_n}f(X^n(\lambda))\ G_T^n(\lambda)\ d\lambda\times 1_{\Lambda^n(r)}\right]\end{aligned} \tag{2.22}$$

Using the first step, we substitute $y = X^n(\lambda)$, and we obtain:

$$E(f(X)) \geqslant \frac{1}{2} E\left[\int_{]X-\alpha,\, X+\alpha[} f(y) \frac{G_T^n(\{X^n\}^{-1}(y))}{|(\partial/\partial\lambda)\, X^n(\{X^n\}^{-1}(y))|}\, dy \times 1_{\Lambda^n(r)}\right]$$
$$\geqslant \int_{\mathbb{R}} f(y)\, E\left[\frac{1}{2}\varphi(|X-y|)\left(1 \wedge \frac{G_T^n(\{X^n\}^{-1}(y))}{|(\partial/\partial\lambda)\, X^n(\{X^n\}^{-1}(y))|}\right) \times 1_{\Lambda^n(r)}\right] dy \tag{2.23}$$

where φ is a continuous function on $\mathbb{R}^+$ such that $1_{[0,\, r]} \leqslant \varphi \leqslant 1_{[0,\, \alpha]}$. We set

$$\theta_n(y) = E\left[\frac{1}{2}\varphi(|X-y|)\left(1 \wedge \frac{G_T^n(\{X^n\}^{-1}(y))}{|(\partial/\partial\lambda)\, X^n(\{X^n\}^{-1}(y))|}\right) \times 1_{\Lambda^n(r)}\right] \tag{2.24}$$

Step 3. On one hand, it is clear that $\theta_n(y_0) > 0$ (recall the definition of $\Lambda^n(r)$, recall that G_T^n is strictly positive, and that $P(\Lambda^n(r)) > 0$). On the other hand, one can show by using the Lebesgue Theorem and Lemma 2.2 that θ_n is continuous. We can easily conclude, by using the continuity of p, and the fact that for all $f \in C_b^+(\mathbb{R})$,

$$\int_{\mathbb{R}} f(y)\, p(y)\, dy \geqslant \int_{\mathbb{R}} f(y)\, \theta_n(y)\, dy \tag{2.25}$$

We at last state a usefull remark.

Remark 2.5. If X is a real valued random variable on Ω, admitting a continuous density p with respect to the Lebesgue measure on $\mathbb{R}$, and if for all $y \in \operatorname{supp} P \circ X^{-1}$, $p(y) > 0$, then p is strictly positive on $\mathbb{R}$.

Proof. Since the support of the law of X is a closed set, we see that for $y \in \partial\{\operatorname{supp} P \circ X^{-1}\}$, $p(y) > 0$. Assume that $(\operatorname{supp} P \circ X^{-1})^c \neq \emptyset$. Then there exists $\{y_k\} \subset (\operatorname{supp} P \circ X^{-1})^c$ such that $y_k \to y \in \partial\{\operatorname{supp} P \circ X^{-1}\}$. Since p is continuous, we deduce that $p(y) = 0$. Thus $(\operatorname{supp} P \circ X^{-1})^c = \emptyset$, and the proof is finished.

In order to prove Theorem 1.5, we will of course apply the previous criterion. In fact, we will only prove that $f(T, \cdot)$ is strictly positive on $\mathbb{R}$, which suffices since T has been arbitrarily fixed. In the next section, we will consider a fixed perturbation v_n, and we will compute $V_t^n(\lambda)$ and its derivatives for any $t \in [0, T]$. Section 4 is devoted to the explicit choice of the sequence v_n of perturbations. In Section 5, we will prove (for some constant $\varepsilon > 0$) that

$$\lim_{n \to \infty} P\left(\left| \frac{\partial}{\partial \lambda} V_T^n(0) \right| \geqslant \varepsilon \right) = 1 \tag{2.26}$$

At last, we will check in Section 6 that for some constant K,

$$\lim_{n \to \infty} P\left(\sup_{|\lambda| \leqslant 1} \left| \frac{\partial}{\partial \lambda} V_T^n(\lambda) \right| + \left| \frac{\partial^2}{\partial \lambda^2} V_T^n(\lambda) \right| \leqslant K \right) = 1 \tag{2.27}$$

Since for all $y_0 \in \operatorname{supp} P \circ V_T^{-1}$, for all $r > 0$, $P(V_T \in]\, y_0 - r,\ y_0 + r[\,) > 0$, we will easily conclude in Section 7.

3. DIFFERENTIABILITY OF THE PERTURBED PROCESS

In this section, we consider a fixed perturbation v_n. We compute $V_t^n(\lambda) = V_t \circ S_n^\lambda$, and we prove that for each t in $[0, T]$, this function is twice differentiable on $[-1, 1]$.

3.1. The Perturbed Process

Recalling that b is defined by (1.6), that $|\cos \theta - 1| \leqslant \theta^2$, and that (1.3) is satisfied, one can easily check that Eq. (1.9) can be written:

$$\begin{aligned} V_t = V_0 &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \theta - 1)\, V_{s-}\, N(d\theta\, d\alpha\, ds) \\ &- \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \theta)\, W_{s-}(\alpha)\, \tilde{N}(d\theta\, d\alpha\, ds) \end{aligned} \tag{3.1}$$

Hence, the perturbed process satisfies

$$\begin{aligned} V_t^n(\lambda) = V_0 &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \theta - 1)\, V_{s-}^n(\lambda)\, N_0^{\lambda, n}(d\theta\, d\alpha\, ds) \\ &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \theta - 1)\, V_{s-}^n(\lambda)\, N_1(d\theta\, d\alpha\, ds) \\ &- \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \theta)\, W_{s-}(\alpha) [N_0^{\lambda, n}(d\theta\, d\alpha\, ds) - \beta_0(\theta)\, d\theta\, d\alpha\, ds] \\ &- \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \theta)\, W_{s-}(\alpha)\, \tilde{N}_1(d\theta\, d\alpha\, ds) \end{aligned} \tag{3.2}$$

But

$$-\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \theta)\, W_{s-}(\alpha)[N_0^{\lambda, n}(d\theta\, d\alpha\, ds) - \beta_0(\theta)\, d\theta\, d\alpha\, ds]$$

$$= -\int_0^t \int_0^1 \int_{-\pi}^{\pi} \sin \gamma_n^\lambda(s, \theta, \alpha)\, W_{s-}(\alpha)\, \tilde{N}_0(d\theta\, d\alpha\, ds)$$

$$-\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \gamma_n^\lambda(s, \theta, \alpha) - \sin \theta)\, W_{s-}(\alpha)\, \beta_0(\theta)\, d\theta\, d\alpha\, ds$$

$$= -\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \theta)\, W_{s-}(\alpha)\, \tilde{N}_0(d\theta\, d\alpha\, ds)$$

$$-\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \gamma_n^\lambda(s, \theta, \alpha) - \sin \theta)\, W_{s-}(\alpha)\, N_0(d\theta\, d\alpha\, ds) \tag{3.3}$$

We finaly obtain:

$$V_t^n(\lambda) = V_0 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \gamma_n^\lambda(s, \theta, \alpha) - 1)\, V_{s-}^n(\lambda)\, N_0(d\theta\, d\alpha\, ds)$$

$$+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \theta - 1)\, V_{s-}^n(\lambda)\, N_1(d\theta\, d\alpha\, ds)$$

$$-\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \theta)\, W_{s-}(\alpha)\, \tilde{N}(d\theta\, d\alpha\, ds)$$

$$-\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \gamma_n^\lambda(s, \theta, \alpha) - \sin \theta)\, W_{s-}(\alpha)\, N_0(d\theta\, d\alpha\, ds) \tag{3.4}$$

3.2. A Lipschitz Property

We study here the continuity of the map $\lambda \mapsto V_t^n(\lambda)$, which will be useful to study its differentiability. We set $U_t^n(\lambda, \mu) = V_t^n(\lambda) - V_t^n(\mu)$. This process satisfies:

$$U_t^n(\lambda, \mu) = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \gamma_n^\lambda(s, \theta, \alpha) - 1)\, U_{s-}^n(\lambda, \mu)\, N_0(d\theta\, d\alpha\, ds)$$

$$+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \theta - 1)\, U_{s-}^n(\lambda, \mu)\, N_1(d\theta\, d\alpha\, ds)$$

$$+\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \gamma_n^\lambda(s, \theta, \alpha) - \cos \gamma_n^\mu(s, \theta, \alpha)) \, V_{s-}^n(\mu) \, N_0(d\theta \, d\alpha \, ds)$$

$$-\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\sin \gamma_n^\lambda(s, \theta, \alpha) - \sin \gamma_n^\mu(s, \theta, \alpha)) \, W_{s-}(\alpha) \, N_0(d\theta \, d\alpha \, ds) \tag{3.5}$$

This equation is a linear S.D.E. If we set

$$K_t^n(\lambda) = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \gamma_n^\lambda(s, \theta, \alpha) - 1) \, N_0(d\theta \, d\alpha \, ds)$$

$$+\int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \theta - 1) \, N_1(d\theta \, d\alpha \, ds) \tag{3.6}$$

then we can write (see Jacod[12]):

$$U_t^n(\lambda, \mu) = \mathscr{E}(K^n(\lambda))_t \int_0^t \int_0^1 \int_{-\pi}^{\pi} \mathscr{E}(K^n(\lambda))_{s-}^{-1} \times \frac{1}{\cos \gamma_n^\lambda(s, \theta, \alpha)}$$

$$\times \{ V_{s-}^n(\mu)[\cos \gamma_n^\lambda(s, \theta, \alpha) - \cos \gamma_n^\mu(s, \theta, \alpha)]$$

$$- W_{s-}(\alpha)[\sin \gamma_n^\lambda(s, \theta, \alpha) - \sin \gamma_n^\mu(s, \theta, \alpha)]\} \, N_0(d\theta \, d\alpha \, ds) \tag{3.7}$$

where the Doléans-Dade exponential is given by (see Jacod and Shiryaev[13]):

$$\mathscr{E}(K^n(\lambda))_t = e^{K_t^n(\lambda)} \prod_{0 \leqslant u \leqslant t} (1 + \Delta K_u^n(\lambda)) \, e^{-\Delta K_u^n(\lambda)}$$

$$= \prod_{0 \leqslant u \leqslant t} (1 + \Delta K_u^n(\lambda)) \tag{3.8}$$

But since any cosinus is in $[-1, 1]$, it is clear that for all $s \leqslant t$,

$$|\mathscr{E}(K^n(\lambda))_t \, \mathscr{E}(K^n(\lambda))_{s-}^{-1}| = \prod_{s \leqslant u \leqslant t} |1 + \Delta K_u^n(\lambda)| \leqslant 1 \tag{3.9}$$

Furthermore, since $|\gamma_n^\lambda| \leqslant \theta_0 < \pi/2$, we see that

$$\left| \frac{1}{\cos \gamma_n^\lambda(s, \theta, \alpha)} \right| \leqslant \frac{1}{\cos \theta_0} < \infty \tag{3.10}$$

At last, since $|\gamma_n^\lambda(s,\theta,\alpha)| \leqslant \frac{3}{2}|\theta|$,

$$\begin{aligned} |\cos \gamma_n^\lambda(s,\theta,\alpha) - \cos \gamma_n^\mu(s,\theta,\alpha)| &\leqslant \tfrac{3}{2}|\theta| \times |\lambda-\mu| \times |v_n(s,\theta,\alpha)| \\ |\sin \gamma_n^\lambda(s,\theta,\alpha) - \sin \gamma_n^\mu(s,\theta,\alpha)| &\leqslant |\lambda-\mu| \times |v_n(s,\theta,\alpha)| \end{aligned} \tag{3.11}$$

Hence, if

$$\begin{aligned} Y_t^n(\lambda) = \frac{1}{\cos\theta_0} \int_0^t \int_0^1 \int_{-\pi}^{\pi} \left[\frac{3}{2} |\theta| \times |V_{s-}^n(\lambda)| + |W_{s-}(\alpha)| \right] \\ \times |v_n(s,\theta,\alpha)| \, N_0(d\theta \, d\alpha \, ds) \end{aligned} \tag{3.12}$$

then for all λ, μ, $|U_t^n(\lambda,\mu)| \leqslant |\lambda-\mu| \times Y_t^n(\lambda)$. In particular, this yields that for all λ,

$$|V_t^n(\lambda)| \leqslant |V_t| + |U_t^n(\lambda, 0)| \leqslant |V_t| + Y_t^n(0) \tag{3.13}$$

Finaly, if

$$\begin{aligned} X_t^n = \frac{1}{\cos\theta_0} \int_0^t \int_0^1 \int_{-\pi}^{\pi} \left[\frac{3}{2} |\theta| \times |V_{s-}| + \frac{3}{2} |\theta| \times Y_{s-}^n(0) + |W_{s-}(\alpha)| \right] \\ \times |v_n(s,\theta,\alpha)| \, N_0(d\theta \, d\alpha \, ds) \end{aligned} \tag{3.14}$$

then for all λ, μ,

$$|U_t^n(\lambda,\mu)| \leqslant |\lambda-\mu| \times X_t^n \tag{3.15}$$

Since we know from Theorem 1.4 that

$$E(\sup_{[0,T]} V_t^2) = E_\alpha(\sup_{[0,T]} W_t^2) < \infty \tag{3.16}$$

we deduce that (recall that $|v_n(s,\theta,\alpha)| \leqslant \eta_n \in L^1(\beta_0(\theta)\, d\theta)$):

$$\begin{aligned} &E(\sup_{[0,T]} |Y_t^n(0)|) \\ &\quad \leqslant \frac{1}{\cos\theta_0} \int_0^T \int_0^1 \int_{-\theta_0}^{\theta_0} \left[\frac{3}{2} |\theta| \eta_n(\theta) \, E(|V_s|) + |W_s(\alpha)| \, \eta_n(\theta) \right] \beta_0(\theta) \, d\theta \, d\alpha \, ds \\ &\quad \leqslant K \int_{-\theta_0}^{\theta_0} \eta_n(\theta) \, \beta_0(\theta) \, d\theta \times E(\sup_{[0,T]} |V_t|) \\ &\qquad + K \int_{-\theta_0}^{\theta_0} \eta_n(\theta) \, \beta_0(\theta) \, d\theta \times E_\alpha(\sup_{[0,T]} |W_t|) < \infty \end{aligned} \tag{3.17}$$

and, by using exactly the same computation,

$$E\big(\sup_{[0,T]} |X_t^n|\big) < \infty \tag{3.18}$$

Thus X_t^n is a.s. finished on $[0, T]$, and we can say that $V_t^n(\lambda)$ satisfies a Lipschitz property on $[-1, 1]$ (for each t).

3.3. Differentiability

We set (for the moment, this is just a notation):

$$\begin{aligned}\frac{\partial}{\partial\lambda} V_t^n(\lambda) = &-\mathscr{E}(K^n(\lambda))_t \int_0^t \int_0^1 \int_{-\pi}^{\pi} \mathscr{E}(K^n(\lambda))_{s-}^{-1} \times \frac{1}{\cos \gamma_n^\lambda(s, \theta, \alpha)} \\ &\times \{ V_{s-}^n(\lambda) \sin \gamma_n^\lambda(s, \theta, \alpha) + W_{s-}(\alpha) \cos \gamma_n^\lambda(s, \theta, \alpha)\} \\ &\times v_n(s, \theta, \alpha)\, N_0(d\theta\, d\alpha\, ds)\end{aligned} \tag{3.19}$$

We obtained this expression by differentiating formaly (3.4), and by using the same argument as in (3.7).

We set $D_t^n(\lambda, \mu) = V_t^n(\mu) - V_t^n(\lambda) - (\mu - \lambda)(\partial/\partial\lambda)\, V_t^n(\lambda)$. Let us compute $D_t^n(\lambda, \mu)$:

$$\begin{aligned}D_t^n(\lambda, \mu) = \mathscr{E}(K^n(\lambda))_t &\int_0^t \int_0^1 \int_{-\pi}^{\pi} \mathscr{E}(K^n(\lambda))_{s-}^{-1} \times \frac{1}{\cos \gamma_n^\lambda(s, \theta, \alpha)} \\ &\times \{ V_{s-}^n(\mu) \times [\cos \gamma_n^\mu(s, \theta, \alpha) - \cos \gamma_n^\lambda(s, \theta, \alpha) \\ &+ (\mu - \lambda) \sin \gamma_n^\lambda(s, \theta, \alpha)\, v_n(s, \theta, \alpha)] \\ &+ U_{s-}^n(\lambda, \mu)(\mu - \lambda) \sin \gamma_n^\lambda(s, \theta, \alpha)\, v_n(s, \theta, \alpha) \\ &- W_{s-}(\alpha) \times [\sin \gamma_n^\mu(s, \theta, \alpha) - \sin \gamma_n^\lambda(s, \theta, \alpha) \\ &- (\mu - \lambda) \cos \gamma_n^\lambda(s, \theta, \alpha)\, v_n(s, \theta, \alpha)]\}\, N_0(d\theta\, d\alpha\, ds)\end{aligned} \tag{3.20}$$

Then a simple computation using Eqs. (3.9), (3.10), (3.15), and something like (3.11) shows that if

$$\begin{aligned}S_t^n = \frac{1}{\cos\theta_0} \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Big[&(|V_{s-}| + X_{s-}^n)\, v_n^2(s, \theta, \alpha) + \frac{3}{2}\, |\theta| \times |v_n(s, \theta, \alpha)| \times X_{s-}^n \\ &+ \frac{3}{2}\, |\theta| \times |W_{s-}(\alpha)| \times v_n^2(s, \theta, \alpha) \Big]\, N_0(d\theta\, d\alpha\, ds)\end{aligned} \tag{3.21}$$

then for all λ, μ,

$$|D_t^n(\lambda, \mu)| \leqslant (\lambda - \mu)^2 \times S_t^n \tag{3.22}$$

Using Eqs. (3.16), (3.18), and the fact that

$$\begin{aligned} &v_n^2(s, \theta, \alpha) + |\theta| \times |v_n(s, \theta, \alpha)| + |\theta| \times v_n^2(s, \theta, \alpha) \\ &\quad \leqslant (\tfrac{1}{2} + \pi + \tfrac{1}{2}\pi)\, \eta_n(\theta) \in L^1(\beta_0(\theta)\, d\theta) \end{aligned} \tag{3.23}$$

we see that

$$E(\sup_{[0, T]} |S_t^n|) < \infty \tag{3.24}$$

It is thus clear that $V_t^n(\lambda)$ is differentiable on $[-1, 1]$, and that its derivative is $(\partial/\partial\lambda)\, V_t^n(\lambda)$.

3.4. Second Differentiability

One can check in the same way that $(\partial/\partial\lambda)\, V_t^n(\lambda)$ is differentiable, and that its derivative is given by

$$\begin{aligned} \frac{\partial^2}{\partial\lambda^2} V_t^n(\lambda) = &\, \mathscr{E}(K^n(\lambda))_t \int_0^t \int_0^1 \int_{-\pi}^{\pi} \mathscr{E}(K^n(\lambda))_{s-}^{-1} \times \frac{1}{\cos \gamma_n^\lambda(s, \theta, \alpha)} \\ &\times \Big\{ -2\sin \gamma_n^\lambda(s, \theta, \alpha) \frac{\partial}{\partial\lambda} V_{s-}^n(\lambda) \times v_n(s, \theta, \alpha) \\ &- V_{s-}^n(\lambda) \cos \gamma_n^\lambda(s, \theta, \alpha)\, v_n^2(s, \theta, \alpha) \\ &+ W_{s-}(\alpha) \sin \gamma_n^\lambda(s, \theta, \alpha)\, v_n^2(s, \theta, \alpha) \Big\}\, N_0(d\theta\, d\alpha\, ds) \end{aligned} \tag{3.25}$$

3.5. Upper Bounds

We will soon use the following equations:

$$\left| \frac{\partial}{\partial\lambda} V_t^n(\lambda) \right| \leqslant R_t^n \tag{3.26}$$

where

$$R_t^n = \frac{1}{\cos\theta_0} \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Bigg[(|V_{s-}| + Y_{s-}^n(0)) \times \frac{3}{2} |\theta| \times |v_n(s,\theta,\alpha)| + |W_{s-}(\alpha)| \times |v_n(s,\theta,\alpha)| \Bigg] N_0(d\theta\, d\alpha\, ds) \tag{3.27}$$

and

$$\left| \frac{\partial^2}{\partial\lambda^2} V_t^n(\lambda) \right| \leqslant \Gamma_t^n \tag{3.28}$$

where

$$\Gamma_t^n = \frac{1}{\cos\theta_0} \int_0^t \int_0^1 \int_{-\pi}^{\pi} \Bigg[3\, |\theta| \times R_{s-}^n \times |v_n(s,\theta,\alpha)| + (|V_{s-}| + Y_{s-}^n(0)) \times v_n^2(s,\theta,\alpha) + |W_{s-}(\alpha)| \times \frac{3}{2} |\theta| \times v_n^2(s,\theta,\alpha) \Bigg] N_0(d\theta\, d\alpha\, ds) \tag{3.29}$$

4. CHOICE OF THE SEQUENCE OF PERTURBATIONS

Recall that

$$\frac{\partial}{\partial\lambda} V_T^n(0) = -\mathscr{E}(K)_T \int_0^T \int_0^1 \int_{-\pi}^{\pi} \mathscr{E}(K)_{s-}^{-1} \times \frac{1}{\cos\theta} \times \{ V_{s-} \sin\theta + W_{s-}(\alpha) \cos\theta \}\, v_n(s,\theta,\alpha)\, N_0(d\theta\, d\alpha\, ds) \tag{4.1}$$

where $K_t = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos\theta - 1)\, N(d\theta\, d\alpha\, ds)$.

The problem is now to choose v_n in such a way that for some $\varepsilon > 0$, some $K < \infty$, the probability

$$P\left(\frac{\partial}{\partial\lambda} V_T^n(0) \in [\varepsilon, K] \right)$$

goes to 1. First, we have to get rid of the random terms $\mathscr{E}(K)_T$ and $\mathscr{E}(K)_{s-}^{-1}$ in (4.1). To this end, we choose $v_n(s,\theta,\alpha)$ equal to 0 for $s \leqslant T - a_n$, for some sequence a_n decreasing to 0, and we use the a.s. continuity of $\mathscr{E}(K)$

at T. Then we notice that the dominant term in $V_{s-} \sin\theta + W_{s-}(\alpha) \cos\theta$ is $W_{s-}(\alpha)\cos\theta$. We thus choose $v_n(s, \theta, \alpha)$ equal to 0 for $|\theta| \leqslant 1/n$ (in order that $|v_n| \in L^1(\beta_0(\theta)\, d\theta)$) and equal to $k\,|\theta|$ for $|\theta| \in [2/n, \theta_1]$ for some $k > 0$ and $\theta_1 \leqslant \theta_0$. This way,

$$\int_{T-a_n}^{T} \int_0^1 \int_{-\pi}^{\pi} |\sin\theta| \times |v_n(s, \theta, \alpha)|\, N_0(d\theta\, d\alpha\, ds)$$

will go to 0, but if a_n is well-chosen, since from (SP), $\theta \notin L^1(\beta_0(\theta)\, d\theta)$,

$$\int_{T-a_n}^{T} \int_0^1 \int_{-\pi}^{\pi} |v_n(s, \theta, \alpha)|\, N_0(d\theta\, d\alpha\, ds)$$

will go to infinity. Of course, this is not satisfying, but a stopping times will allow us to "cutoff" this second integral.

Let us now be precise. First, let us recall a lemma that can be found in Graham and Méléard,[11] p. 15.

Lemma 4.1. Assume (H)-1. There exists $0 < c < C < \infty$ and $q > 0$ such that for all $t \in [0, T]$,

$$P_\alpha(c \leqslant |W_t| \leqslant C) \geqslant q \tag{4.2}$$

We will also need the following lemma

Lemma 4.2. One can build a sequence ϕ_n of positive, even, C^1 functions on $[-\theta_0, \theta_0]$ such that $\phi_n(-\theta_0) = \phi_n(\theta_0) = 0$, such that $\phi_n(\theta) \leqslant k\,|\theta|$ for some $k \leqslant 1/2$, such that if

$$\xi_n(\theta) = |\phi'_n(\theta)| + r 2^{r+2} \frac{\phi_n(\theta)}{|\theta|} \tag{4.3}$$

then $\xi_n \in L^1(\beta_0(\theta)\, d\theta)$ and $\xi_n \leqslant 1/2$, and such that there exists a sequence a_n decreasing to 0 satisfying

$$a_n \int_{-\theta_0}^{\theta_0} \phi_n(\theta)\, \beta_0(\theta)\, d\theta \longrightarrow \infty \tag{4.4}$$

$$a_n \int_{-\theta_0}^{\theta_0} (|\theta| \phi_n(\theta) + \phi_n^2(\theta))\, \beta_0(\theta)\, d\theta \longrightarrow 0 \tag{4.5}$$

Proof. We clearly can build a sequence ϕ_n of even, positive and C^1 functions such that, for some $k \in]0, 1/2]$, $\phi_n(\theta) \leqslant k\,|\theta|$, such that

$$\phi_n(\theta) = \begin{cases} 0 & \text{if} \quad |\theta| \leqslant 1/n \\ k\,|\theta| & \text{if} \quad |\theta| \in [2/n, \theta_0/2(1+k)] \\ 0 & \text{if} \quad |\theta| \in [\theta_0/(1+k), \theta_0] \end{cases} \tag{4.6}$$

and such that

$$|\phi_n'(\theta)| \leqslant \begin{cases} 0 & \text{if} \quad |\theta| \leqslant 1/n \\ 4k & \text{if} \quad |\theta| \in [1/n, 2/n] \\ k & \text{if} \quad |\theta| \in [2/n, \theta_0/2(1+k)] \\ 2k & \text{if} \quad |\theta| \in [\theta_0/2(1+k), \theta_0/(1+k)] \\ 0 & \text{if} \quad |\theta| \in [\theta_0/(1+k), \theta_0] \end{cases} \tag{4.7}$$

Then ξ_n is bounded, and vanishes near 0, it thus is in $L^1(\beta_0(\theta)\,d\theta)$. Furthermore, $\xi_n \leqslant 4k + r2^{r+2}k$, which is smaller than 1/2 if we choose k small enough. We now choose

$$a_n = \left(\int_{-\theta_0}^{\theta_0} \phi_n(\theta)\,\beta_0(\theta)\,d\theta \right)^{-1/2} \tag{4.8}$$

We see that

$$\begin{aligned} \int_{-\theta_0}^{\theta_0} \phi_n(\theta)\,\beta_0(\theta)\,d\theta &\geqslant 2 \int_{2/n}^{\theta_0/2(1+k)} k\theta \times \frac{k_0}{\theta^r}\,d\theta \\ &= 2kk_0 \int_{2/n}^{\theta_0/2(1+k)} \theta^{1-r}\,d\theta \end{aligned} \tag{4.9}$$

goes to infinity when n goes to infinity, since r is greater than 2. Hence a_n goes to 0, and condition (4.4) is satisfied. On the other hand,

$$a_n \int_{-\theta_0}^{\theta_0} (|\theta|\,\phi_n(\theta) + \phi_n^2(\theta))\,\beta_0(\theta)\,d\theta \leqslant K a_n \int_0^{\theta_0} \theta^{2-r}\,d\theta \leqslant K a_n \tag{4.10}$$

which goes to 0 since $r < 3$. The lemma is proved.

We now define a stopping time that will allow the derivative at 0 not to be too large. Consider the following process:

$$Z_t^n = \int_0^t \int_{c \leqslant |W_s(\alpha)| \leqslant C} \int_{-\pi}^{\pi} \phi_n(\theta)\,N_0(d\theta\,d\alpha\,ds) \tag{4.11}$$

We fix $l>0$, and we set

$$T_n=\inf\{t>T-a_n/Z^n_t-Z^n_{T-a_n}\geqslant l\} \tag{4.12}$$

We now can define our sequence of perturbations ($\mathrm{sg}(x)$ denotes the signe of x).

$$v_n(s,\theta,\alpha)=1_{[T-a_n,\,T_n\wedge T]}(s)\,1_{\{c\leqslant|W_{s-}(\alpha)|\leqslant C\}}\,\mathrm{sg}(\mathscr{E}(K)_{s-})\,\mathrm{sg}(W_{s-}(\alpha))\,\phi_n(\theta) \tag{4.13}$$

For each n, v_n is a perturbation (see Definition 2.1), since it is predictable, and since it satisfies (2.1)–(2.3) thanks to Lemma 4.2.

We at last prove the essential following convergence:

$$\lim_{n\to\infty} P(T_n<T)=1 \tag{4.14}$$

Indeed,

$$\begin{aligned}
P(T_n<T)&\geqslant P(Z^n_T-Z^n_{T-a_n}\geqslant l)\geqslant 1-e^l E(e^{-(Z^n_T-Z^n_{T-a_n})})\\
&\geqslant 1-e^l\exp\left\{-\int_{T-a_n}^{T}\int_{c\leqslant|W_{s-}(\alpha)|\leqslant C}\int_{-\pi}^{\pi}(1-e^{-\phi_n(\theta)})\,\beta_0(\theta)\,d\theta\,d\alpha\,ds\right\}\\
&\geqslant 1-e^l\exp\left\{-a_n\times q\times\frac{1}{2}\int_{-\pi}^{\pi}\phi_n(\theta)\,\beta_0(\theta)\,d\theta\right\}
\end{aligned} \tag{4.15}$$

which goes to 1 thanks to Eq. (4.4). We have used Lemma 4.1 and the fact that since ϕ_n is smaller than 1, $1-e^{-\phi_n}\geqslant\phi_n/2$.

5. THE DERIVATIVE AT 0 IS LARGE ENOUGH

Thanks to our choice for the perturbation v_n, we can write

$$\begin{aligned}
&\left|\frac{\partial}{\partial\lambda}V^n_T(0)\right|\\
&\quad=|\mathscr{E}(K)_T|\times\left|\int_{T-a_n}^{T_n\wedge T}\int_{c\leqslant|W_{s-}(\alpha)|\leqslant C}\int_{-\theta_0}^{\theta_0}|\mathscr{E}(K)^{-1}_{s-}|\right.\\
&\qquad\left.\times\{(\tan\theta)\,V_{s-}\,\mathrm{sg}(W_{s-}(\alpha))+|W_{s-}(\alpha)|\}\,\phi_n(\theta)\,N_0(d\theta\,d\alpha\,ds)\right|
\end{aligned}$$

$$\geqslant |\mathscr{E}(K)_T| \int_{T-a_n}^{T_n \wedge T} \int_{c \leqslant |W_{s-}(\alpha)| \leqslant C} \int_{-\theta_0}^{\theta_0} |\mathscr{E}(K)_{s-}^{-1}| \times c \times \phi_n(\theta)\, N_0(d\theta\, d\alpha\, ds)$$

$$- |\mathscr{E}(K)_T| \int_{T-a_n}^{T} \int_0^1 \int_{-\theta_0}^{\theta_0} |\mathscr{E}(K)_{s-}^{-1}| \times |\tan\theta| \times |V_{s-}| \times \phi_n(\theta)\, N_0(d\theta\, d\alpha\, ds)$$

$$\geqslant A_n - B_n \tag{5.1}$$

First A_n is larger than

$$\inf_{[T-a_n,\, T]} |\mathscr{E}(K)_T\, \mathscr{E}(K)_{s-}^{-1}| \times c \times (Z^n_{T \wedge T_n} - Z^n_{T-a_n}) \tag{5.2}$$

But $\mathscr{E}(K)$ is a.s. continuous (and does not vanish) at T, thus the first term in the product goes a.s. to 1. Furthermore, using Eqs. (4.12) and (4.14), we see that

$$\lim_{n \to \infty} P(Z^n_{T \wedge T_n} - Z^n_{T-a_n} \geqslant l) = 1 \tag{5.3}$$

It is thus clear that

$$\lim_{n \to \infty} P(A_n \geqslant cl/2) = 1 \tag{5.4}$$

On the other hand,

$$B_n \leqslant \sup_{[T-a_n,\, T]} |\mathscr{E}(K)_T\, \mathscr{E}(K)_{s-}^{-1}| \times \frac{1}{\cos\theta_0}$$

$$\times \int_{T-a_n}^{T} \int_0^1 \int_{-\theta_0}^{\theta_0} |V_{s-}| \times |\theta|\, \phi_n(\theta)\, N_0(d\theta\, d\alpha\, ds) \tag{5.5}$$

First, we have already seen (see Eq. (3.9)) that the first term in the product is always smaller than 1. The last term goes to 0 in L^1, thanks to (4.5) and (3.16), since

$$E\left[\int_{T-a_n}^{T} \int_0^1 \int_{-\theta_0}^{\theta_0} |V_{s-}| \times |\theta|\, \phi_n(\theta)\, N_0(d\theta\, d\alpha\, ds)\right]$$

$$\leqslant E(\sup_{[0,\, T]} |V_{s-}|) \times a_n \int_{-\theta_0}^{\theta_0} |\theta|\, \phi_n(\theta)\, \beta_0(\theta)\, d\theta \tag{5.6}$$

Hence B_n goes to 0 in probability, and we finaly deduce that

$$\lim_{n \to \infty} P\left(\left|\frac{\partial}{\partial \lambda} V_T^n(0)\right| \geqslant cl/4\right) = 1 \tag{5.7}$$

The first part of our criterion is satisfied.

6. THE DERIVATIVES ARE NOT TOO LARGE

We still have to check that there exists $K < \infty$ such that

$$P\left(\sup_{|\lambda| \leqslant 1} \left|\frac{\partial}{\partial \lambda} V_T^n(\lambda)\right| \leqslant K\right) \longrightarrow 1 \tag{6.1}$$

and

$$P\left(\sup_{|\lambda| \leqslant 1} \left|\frac{\partial^2}{\partial \lambda^2} V_T^n(\lambda)\right| \leqslant K\right) \longrightarrow 1 \tag{6.2}$$

We refer to Section 3. for the notations. In order to prove (6.1), we just have to check that $P(R_T^n \leqslant K)$ goes to 1 (see (3.26) and (3.27)). First, we will need the following preliminary estimation (L is a constant independant of n):

$$E[\sup_{[0,\,T]} Y_t^n(0)] \leqslant L \tag{6.3}$$

But (M is a constant)

$$\begin{aligned}
&E[\sup_{[0,\,T]} Y_t^n(0)]\\
&\leqslant ME\left[\int_{T-a_n}^{T_n \wedge T} \int_{c \leqslant |W_{s-}(\alpha)| \leqslant C} \int_{-\pi}^{\pi} [|\theta|\,|V_{s-}| + |W_{s-}(\alpha)|]\, \phi_n(\theta)\, N_0(d\theta\, d\alpha\, ds)\right]\\
&\leqslant ME\left[\int_{T-a_n}^{T} \int_0^1 \int_{-\theta_0}^{\theta_0} |\theta|\,|V_{s-}|\, \phi_n(\theta)\, N_0(d\theta\, d\alpha\, ds)\right]\\
&\quad + ME\left[\int_{T-a_n}^{T_n} \int_{c \leqslant |W_{s-}(\alpha)| \leqslant C} \int_{-\pi}^{\pi} \phi_n(\theta)\, N_0(d\theta\, d\alpha\, ds)\right]
\end{aligned} \tag{6.4}$$

Thanks to the definition (4.12) of T_n, the second term is smaller than $M(l+\|\phi_n\|_\infty)$. But ϕ_n is always smaller than 1/2, and thus the second term is smaller than $M(l+1/2)$. On the other hand, the first term is smaller than

$$M\int_{T-a_n}^{T}\int_0^1\int_{-\theta_0}^{\theta_0} E(|V_{s-}|)\times|\theta|\;\phi_n(\theta)\;\beta_0(\theta)\;d\theta\;d\alpha\;ds$$

$$\leqslant ME(\sup_{[0,T]}|V_t|)\times a_n\int_{-\theta_0}^{\theta_0}|\theta|\;\phi_n(\theta)\;\beta_0(\theta)\;d\theta \tag{6.5}$$

which goes to 0, thanks to (4.5) and (3.16). Inequality (6.3) is satisfied.

We now write R_t^n as $(1/\cos\theta_0)(\frac{3}{2}R_t^{n,1}+R_t^{n,2})$, where

$$R_t^{n,1}=\int_0^t\int_0^1\int_{-\pi}^{\pi}(|V_{s-}|+Y_{s-}^n(0))\times|\theta|\times|v_n(s,\theta,\alpha)|\;N_0(d\theta\;d\alpha\;ds) \tag{6.6}$$

$$R_t^{n,2}=\int_0^t\int_0^1\int_{-\pi}^{\pi}|W_{s-}(\alpha)|\times|v_n(s,\theta,\alpha)|\;N_0(d\theta\;d\alpha\;ds) \tag{6.7}$$

It is clear, thanks to the definitions of v_n and T_n, and since $\phi_n\leqslant 1/2$, that $R_T^{n,2}\leqslant C(l+1/2)$. On the other hand, (3.16), (6.3) and (4.5) yield that $R_T^{n,1}$ goes to 0 in L^1. Hence, $P(R_T^n\leqslant 2C(l+1/2))$ goes to 1, and (6.1) is satisfied.

Notice that we have proved in particular that there exists a constant L independant of n such that

$$E(\sup_{[0,T]}R_t^n)\leqslant L \tag{6.8}$$

In order to prove (6.2), we have to check that $P(\Gamma_T^n\leqslant K)$ goes to 1 (recall (3.28) and (3.29)). Thanks to (4.5), (3.16), (6.3), and (6.8), we see that

$$E(\Gamma_T^n)\to 0 \tag{6.9}$$

which gives immediately the result.

Notice that we do not need to choose l (see the definition of T_n, (4.12)): this might look strange, but it in fact is natural. First, if l is large, then the derivative at 0 will be more easily large, but the derivative and second derivative will be less easily bounded on $[-1,1]$. As a second reason, notice that we use a sequence of perturbations that would make explode $(\partial/\partial\lambda)\;V_T^n(0)$ if we did not use T_n.

7. CONCLUSION

We have found some constants $\varepsilon > 0$ and $K < \infty$ such that

$$\lim_{n\to\infty} P\left(\left|\frac{\partial}{\partial\lambda} V^n_T(0)\right| \geqslant \varepsilon;\ \sup_{|\lambda|\leqslant 1} \left|\frac{\partial}{\partial\lambda} V^n_T(\lambda)\right| \leqslant K;\ \sup_{|\lambda|\leqslant 1} \left|\frac{\partial^2}{\partial\lambda^2} V^n_T(\lambda)\right| \leqslant K\right) = 1 \tag{7.1}$$

Let now y_0 be a point of the support of the law of V_T. Then for any $r > 0$,

$$P\left(|V_T - y_0| \leqslant r;\ \left|\frac{\partial}{\partial\lambda} V^n_T(0)\right| \geqslant \varepsilon;\ \sup_{|\lambda|\leqslant 1} \left|\frac{\partial}{\partial\lambda} V^n_T(\lambda)\right| \leqslant K;\ \sup_{|\lambda|\leqslant 1} \left|\frac{\partial^2}{\partial\lambda^2} V^n_T(\lambda)\right| \leqslant K\right)$$

$$\xrightarrow[n\to\infty]{} P(|V_T - y_0| \leqslant r) > 0 \tag{7.2}$$

Theorem 2.3 allows us to say that $f(T, y_0) > 0$. Since we know from Theorem 1.4 that $f(T, \cdot)$ is continuous on $\mathbb{R}$, Remark 2.5 allows us to deduce that for all $y \in \mathbb{R}$, $f(T, y) > 0$. At last, since $T > 0$ has been arbitrarily fixed, this holds for any $T > 0$, and the proof of Theorem 1.5 is finished.

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